Initial data for stationary space-times near space-like infinity

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Abstract

We study Cauchy initial data for asymptotically flat, stationary vacuum space-times near space-like infinity. The fall-off behavior of the intrinsic metric and the extrinsic curvature is characterized. We prove that they have an analytic expansion in powers of a radial coordinate. The coefficients of the expansion are analytic functions of the angles. This result allow us to fill a gap in the proof found in the literature of the statement that all asymptotically flat, vacuum stationary space-times admit an analytic compactification at null infinity.

Stationary initial data are physical important and highly non-trivial examples of a large class of data with similar regularity properties at space-like infinity, namely, initial data for which the metric and the extrinsic curvature have asymptotic expansion in terms of powers of a radial coordinate. We isolate the property of the stationary data which is responsible for this kind of expansion.

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1 Introduction

The far-field behavior of stationary space-times is by now reasonably well understood (see [3] and reference therein). However, all of this information is encoded in terms of quantities intrinsic to the abstract three dimensional manifold $\tilde{X}$ of trajectories of the time like Killing vector ξ^a. In order to extract from this description general properties of asymptotically flat solutions of the field equations, we have to express this information in terms of quantities that can be defined for general asymptotically flat solutions. Take an space like hypersurface $\tilde{S}$ in the space time. Consider the intrinsic metric and the extrinsic curvature of $\tilde{S}$. How are these fields related to the ones defined on the abstract

manifold $\tilde{X}$? And, more important, what is the precise fall off behavior of these fields at space-like infinity? The purpose of this article is to answer these questions. The result is somehow unexpected: in contrast the fields defined in $\tilde{X}$ the fields in the Cauchy initial data $\tilde{S}$ are never analytic at infinity, unless we are dealing with the static case. By 'analytic at infinity' we mean that the tensor components of the fields are analytic with respect to an appropriate Cartesian coordinate system in a neighborhood of infinity. However, they are analytic in terms of a radial coordinate and the corresponding angles. The typical behavior of the fields is represented by the radial coordinate $r = (\sum_{i=1}^{3}(x^j)^2)^{1/2}$ itself, which is not an analytic function of the Cartesian coordinates x^i at the origin. This is precisely the non-analytic type of behavior of the intrinsic metric and the extrinsic curvature near space like infinity in a Cauchy slice. The fall-off of those tensors in a particular foliation is given by theorems 2.6 and 2.7. This constitute our main result. These theorems are essential in order to prove that all stationary, asymptotically flat, space-times admit an analytic conformal compactification of null infinity. We fill in this way a gap in the proof of this result given in [8].

In a previous work, we have studied a class of initial data that have asymptotic expansion in terms of powers of a radial coordinate[7]. In order to construct this class we impose a condition on the square of the conformal extrinsic curvature. In theorem 2.7 we prove that the stationary initial data satisfy the same condition.

2 Initial data for stationary space-time near space like infinity

Let $\tilde{M}, \tilde{g}_{ab}$ be a stationary vacuum space-time. The collection of all trajectories of the time like Killing vector ξ^a defines an abstract manifold $\tilde{X}$, called 'the quotient space'. When ξ^a is surface orthogonal (i.e. when the space-time is static) then $\tilde{X}$ is naturally identified with one of the hypersurfaces of $\tilde{M}$ everywere orthogonal to ξ^a. Each trajectory of ξ^a would intersect the hypersurface in exactly one point. In the non-hypersurface orthogonal case, however, there is no natural way of introducing such surface on $\tilde{M}$. This is the reason why $\tilde{X}$ and not a slice $\tilde{S}$ is the most convenient object to study in the stationary spaces-times [11].

The field equations have a remarkably simple form in terms of quantities defined on $\tilde{X}$. The metric, with signature $+---$, can locally be written as

$$\tilde{g} = \lambda(dt + \beta_i d\tilde{x}^i) - \lambda^{-1}\tilde{\gamma}_{ij}d\tilde{x}^i d\tilde{x}^j \quad (i, j = 1, 2, 3), \tag{1}$$

where λ, β_i, and the Riemannian metric $\tilde{\gamma}_{ij}$ depend only on the spatial coordinates $\tilde{x}^k$. In these coordinates, the Killing vector is given by $\xi^a = (\partial/\partial t)^a$. Note that $\lambda = g_{ab}\xi^a\xi^b$. The metric $\tilde{\gamma}_{ij}$, λ and β_i are naturally defined on $\tilde{X}$.

On the other hand, the metric $\tilde{g}_{ab}$ has a 3+1 decomposition with respect to

the hypersurface $\tilde{S}$, defined by $t = constant$,

$$\tilde{g} = \tilde{N}^2 dt^2 - \tilde{h}_{ij}(N^i dt + d\tilde{x}^i)(N^j dt + d\tilde{x}^j), \tag{2}$$

where $\tilde{N}$ is the lapse function, N^i the shift vector and $\tilde{h}_{ij}$ the intrinsic metric of the slice $\tilde{S}$. We have the following relations between these quantities

$$\tilde{N}^2 = \frac{\lambda}{1 - \lambda^2 \tilde{\beta}^i \beta_i}, \tag{3}$$

$$N_j = \lambda \beta_j, \quad N^j = -\tilde{N}^2 \lambda \tilde{\beta}^j, \tag{4}$$

$$\tilde{h}_{ij} = \lambda^{-1} \tilde{\gamma}_{ij} - \lambda \beta_i \beta_j, \quad \tilde{h}^{ij} = \lambda \tilde{\gamma}^{ij} + \lambda^2 \tilde{N}^2 \beta^i \beta^j, \tag{5}$$

were we have defined

$$\tilde{N}_j = \tilde{h}_{ij} N^i, \quad \tilde{\beta}^j = \tilde{\gamma}^{jk} \beta_k. \tag{6}$$

Note that the indices are moved with different metrics. In general, indexes of tensors defined on $\tilde{X}$ will be moved with the metric $\tilde{\gamma}_{ij}$ and the ones defined on $\tilde{S}$ with $\tilde{h}_{ij}$.

We will write now the vacuum field equations in terms of the quantities intrinsic to $\tilde{X}$. Let the covariant derivative $\tilde{D}_i$ be defined with respect to $\tilde{\gamma}_{ij}$. From the vacuum field equations it follows that the quantity

$$\omega_i = -\lambda^2 \tilde{\epsilon}_{ijk} \tilde{D}^j \beta^k, \tag{7}$$

is curl free, i.e.

$$\tilde{D}_{[i} \omega_{j]} = 0. \tag{8}$$

Where $\tilde{\epsilon}_{ijk} = \tilde{\epsilon}_{[ijk]}$ and $\tilde{\epsilon}_{123} = |\det \tilde{\gamma}_{ij}|^{1/2}$. Thus, there exist a scalar field ω (the twist of the Killing vector ξ^a) such that

$$\tilde{D}_i \omega = \omega_i. \tag{9}$$

By equation (7), the covector β_k is determined only up to a gradient

$$\beta_k \to \beta_k + \partial_k f, \tag{10}$$

where f is a scalar field in $\tilde{X}$. Under this change the metric (1) remains unchanged if we set $t \to t - f$.

It is convenient not to work with the scalar λ and ω but with certain algebraic combinations introduced by Hansen [12]

$$\tilde{\phi}_M = \frac{1}{4\lambda}(\lambda^2 + \omega^2 - 1), \tag{11}$$

$$\tilde{\phi}_S = \frac{1}{2\lambda}\omega, \tag{12}$$

$$\tilde{\phi}_K = \frac{1}{4\lambda}(\lambda^2 + \omega^2 + 1). \tag{13}$$

These functions satisfy the relation

$$\tilde{\phi}_M^2 + \tilde{\phi}_S^2 - \tilde{\phi}_K^2 = -\frac{1}{4}. \tag{14}$$

Denote by $\tilde{\phi}$ any of the functions $\tilde{\phi}_M, \tilde{\phi}_S, \tilde{\phi}_K$. The vacuum field equations then read

$$\tilde{\Delta}\tilde{\phi} = 2\tilde{R}\tilde{\phi} \tag{15}$$

$$\tilde{R}_{ij} = 2(\tilde{D}_i\tilde{\phi}_M\tilde{D}_j\tilde{\phi}_M + \tilde{D}_i\tilde{\phi}_S\tilde{D}_j\tilde{\phi}_S - \tilde{D}_i\tilde{\phi}_K\tilde{D}_j\tilde{\phi}_K), \tag{16}$$

where $\tilde{R}_{ij}$ is the Ricci tensor of $\tilde{\gamma}_{ij}$ and $\Delta = \tilde{D}^i\tilde{D}_i$. Of course, since (14) holds, only two of the three equations (15) are independent.

We assume that $(\tilde{X}, \tilde{\gamma}_{ij}, \tilde{\phi}_M, \tilde{\phi}_S)$ is asymptotically flat in the following sense. There exist a manifold X, consisting of $\tilde{X}$ and an additional point i; such that:

(i) For some real constant $B^2 > 0$ the conformal factor

$$\Omega = \frac{1}{2}B^{-2}[(1 + 4(\tilde{\phi}_M^2 + \tilde{\phi}_S^2))^{1/2} - 1] \tag{17}$$

is C^2 on X and satisfies

$$\Omega(i) = 0, \quad D_i\Omega(i) = 0 \tag{18}$$

at the point i.

(ii) $\gamma_{ij} = \Omega^2\tilde{\gamma}_{ij}$ extends to a $C^{4,\alpha}$ metric on X and

$$D_jD_k\Omega(i) = 2\gamma_{jk}(i). \tag{19}$$

(iii) Ω is $C^{2,\alpha}$ on X.

Define the rescaled functions $\phi = \tilde{\phi}/\sqrt{\Omega}$. The following theorem has been proved in [4], see also [14].

Theorem 2.1 (Beig-Simon). *For any asymptotically flat solution* $(\tilde{\gamma}_{ij}, \tilde{\phi}_M, \tilde{\phi}_S)$ *of equations (15) and (16) there exist a chart defined in some neighborhood of* i *in* X *such that* $(\gamma_{ij}, \phi_M, \phi_S, \Omega)$ *are analytic.*

The asymptotic flatness condition can be written in terms of the decay at infinity of the physical metric $\tilde{g}_{ab}$. It can be relaxed considerable, see [13]. Existence of asymptotically solutions of the field equations, without any further symmetry, has been proved in [16]. Theorem 2.1 make use of the specific conformal factor (17), other choices are also possible, for example the one made in [14].

Given the chart in which γ_{ij} is analytic, we can make a coordinate transformation to γ-normal coordinates x^j centered at i. The fields γ_{ij}, ϕ_M, ϕ_S,

Ω are also analytic with respect to x^j. We define the radial coordinate $r = (\sum_{i=1}^{3}(x^j)^2)^{1/2}$. The first terms in the expansion of ϕ_M and ϕ_S are given by

$$\phi_M = M + O(r), \quad \phi_S = S^i x_i + O(r^2), \tag{20}$$

where M is the total mass, and S^i is the intrinsic angular momentum.

Is important to recall that theorem 2.1 asserts only the analyticity of γ_{ij}, ϕ_M, ϕ_S, Ω, and not of the other quantities, like, for example, λ, ϕ_K, β_i. In fact these quantities are *not* analytic as we will see. However, the non-analyticity of these functions is given only by the function r. According to this, we define the following function space, which is the analytic analog of the spaces defined in [7].

Definition 2.2. *We define the space E^ω as the set $E^\omega = \{f = f_1 + r f_2 : f_1, f_2$ analytic functions in a neighborhood of $i\}$.*

We have the following lemma.

Lemma 2.3. *Let $f, g \in E^\omega$, then*

(i) $f + g \in E^\omega$

(ii) $fg \in E^\omega$

(iii) If $f \neq 0$ *then* $1/f \in E^\omega$

Proof. The first two assertions are obvious, for (iii) see [7]. □

In what follows, the idea is to prove that all the relevant fields essentially belong to E^ω. We begin with a useful lemma regarding the conformal factor (17).

Lemma 2.4. *The conformal factor Ω given by (17) has the following form $\Omega = r^2 f_\Omega$, where f_Ω is an analytic, positive, function, and $f_\Omega(i) = 1$.*

Proof. The conformal factor Ω satisfies equation (16) of [4] (see also equation (2.9) of [14]). This equation has the following form

$$D_i D_j \Omega = \Omega T_{ij} + f_1 \gamma_{ij} + f_2 D_i \Omega D_j \Omega, \tag{21}$$

where the tensor T_{ij} and the functions f_1, f_2 are analytic. We want to prove that the symmetric and trace free part of the tensor $D_{j_1} \cdots D_{j_n} \Omega$, for all n, vanish at the point i. To prove this we use induction on n, in the same way as in the proof of lemma 1 in [2]. The cases $n = 1, 2$ are given by equations (18) and (19). To perform the induction step we assume $n \geq 2$ and show that the statement for $n - 1$ implies that for n. Using equation (21), we express $D_{j_1} \cdots D_{j_n} \Omega$ in terms of $D_{j_1} \cdots D_{j_{n-1}} \Omega$. Using the induction hypothesis the result follows.

We use the previous result in the analytic expansion of Ω

$$\Omega = r^2 + \frac{1}{3!} x^i x^j x^k D_i D_j D_k \Omega|_i + \frac{1}{4!} x^i x^j x^k x^q D_i D_j D_k D_q \Omega|_i \cdots, \tag{22}$$

to conclude that $\Omega = r^2 f_\Omega$, where f_Ω is an analytic, positive, function, and $f_\Omega(i) = 1$. There exist a much more elegant method to prove the same result directly out of equation (21) using complex analysis, as it is explained in [10]. $\square$

From equation (14) we obtain the following relation for the rescaled functions ϕ

$$\phi_M^2 + \phi_S^2 - \phi_K^2 = -\frac{1}{4\Omega}. \tag{23}$$

Then, the function ϕ_K has the form

$$\phi_K = \frac{f_K}{\sqrt{\Omega}}, \quad f_K = \sqrt{\Omega(\phi_M^2 + \phi_S^2) + 1/4}. \tag{24}$$

The function f_K is analytic. We see that ϕ_K is not analytic, in fact it blows up at i. The functions ω and λ are given by

$$\omega = \frac{\phi_S}{(\phi_K - \phi_M)}, \quad \lambda = \frac{1}{2\sqrt{\Omega}(\phi_K - \phi_M)}. \tag{25}$$

From these formula, we see immediately that these functions are also not analytic at i, since $\sqrt{\Omega}$ is not analytic.

From lemma 2.3 and equations (25), (24) it follows that

$$\lambda \in E^\omega, \quad \lambda(i) = 1, \tag{26}$$

and

$$\omega = r\hat{\omega}, \quad \hat{\omega} \in E^\omega, \quad \hat{\omega} = S^i x_i + O(r^2). \tag{27}$$

In order to characterize the behavior of β_i we have to solve equation (7). First we will write this equation in terms of the functions ϕ . We write equation (15) as

$$L_{\tilde{\gamma}}\tilde{\phi} = \frac{15}{8}\tilde{\phi}, \tag{28}$$

where $L_{\tilde{\gamma}} = \tilde{\Delta} - \tilde{R}/8$. We use the formula

$$L_{\tilde{\gamma}}\tilde{\phi} = \Omega^{5/2} L_\gamma \phi, \tag{29}$$

which holds for an arbitrary conformal factor, to obtain

$$L_\gamma \phi = \frac{15}{8}\Omega^{-2}\tilde{R}\phi. \tag{30}$$

Take equation (30) for ϕ_M and for ϕ_S, multiply it by ϕ_S and ϕ_M respectively, and take the difference between this two equations. We obtain

$$\phi_S \Delta\phi_M - \phi_M \Delta\phi_S = 0. \tag{31}$$

From this equation it follows that the vector J_i^1 defined by

$$J_i^1 = \phi_S D_i \phi_M - \phi_M D_i \phi_S, \tag{32}$$

is divergence free $D^i J_i^1 = 0$. In the same way we obtain that the following vectors are divergence free

$$J_i^2 = \phi_M D_i \phi_K - \phi_K D_i \phi_M, \tag{33}$$

$$J_i^3 = \phi_S D_i \phi_K - \phi_K D_i \phi_S. \tag{34}$$

The vector J^1 is analytic, but the vectors J^2 and J^3 are not. In particular, J^3 has the form

$$J_i^3 = \frac{1}{\Omega^{3/2}} \left\{ \Omega^2 (\phi_S D_i f_K - f_K D_i \phi_S) - \frac{1}{2} \phi_S f_K D_i \Omega \right\}. \tag{35}$$

The expression in curly brackets in this equation is analytic.

In terms of the conformal metric γ_{ij} equation (7) is given by

$$\frac{1}{\Omega \lambda^2} D_i \omega = 4(J_i^1 - J_i^3) = -\epsilon_{ijk} D^j \beta^k, \tag{36}$$

where ϵ_{ijk} is the volume element with respect to γ_{ij}, and we have used (25). This equation can be written in terms of the flat metric δ_{ij}, the flat volume element $\bar{\epsilon}_{ijk}$ and partial derivatives ∂_i

$$\bar{\epsilon}^{ijk} \partial_j \beta_k = J^i, \quad J^i = -\frac{4}{\sqrt{|\gamma|}} \gamma^{ij} (J_j^1 - J_j^3). \tag{37}$$

Note that $\partial_i J^i = 0$. We use equation (35) and lemma 2.4 to conclude that the vector J^i, in normal coordinates, has the form

$$J^i = H_1^i + \frac{1}{r^3} (r^2 H_2^i + x^i H), \tag{38}$$

where H_1^i, H_2^i, H are analytic functions. Using (20) we find that

$$H_2^i(0) = S^i. \tag{39}$$

Lemma 2.5. *There exist a solution β_i of equation (36) which, in normal coordinates x^i, has the following form*

$$\beta_i = \beta_i^1 + \frac{\beta_i^2}{r}, \tag{40}$$

where β_i^1 and β_i^2 are analytic functions of x^i given by

$$\beta_i^1 = \bar{\epsilon}_{ijk} f_1^j x^k, \quad \beta_i^2 = \bar{\epsilon}_{ijk} f_2^j x^k, \tag{41}$$

where f_1^i and f_2^i are analytic. In particular, this implies that $\beta_i x^i = 0$.

Proof. We expand in powers series in the coordinates x^i the analytic functions H_1^i, H_2^i, H in the expression (38). For each power, we use the following explicit formula in order to solve (37). Let $p^i_{(m)}$ a three tuple of homogenous polynomials of order m, which satisfies

$$\partial_i(r^s p^i_{(m)}) = 0, \tag{42}$$

for some integer s. Then, we have

$$\bar{\epsilon}^{ijk}\partial_j\beta_k^{(m)} = r^s p^i_{(m)}, \quad \beta_i^{(m)} = r^s(s+m+2)^{-1}\bar{\epsilon}_{ijk}p^j_{(m)}x^k. \tag{43}$$

By (43), the series defined by the $p^i_{(m)}$ majorizes the one defined by the $\beta_k^{(m)}$, hence the last one defines a convergent power series. Note that the term with x^i in (38) made no contribution to β_i. $\square$

We define the conformal factor

$$\hat{\Omega} = \sqrt{\lambda}\Omega, \tag{44}$$

and the rescaled four metric and three metric by $g_{ab} = \hat{\Omega}^2\tilde{g}_{ab}$ and $h_{ij} = \hat{\Omega}^2\tilde{h}_{ij}$. Note that $\hat{\Omega}$ is not analytic. We have

$$g = \Omega^2\lambda^2(dt + \beta_i dx^i)^2 - \gamma_{ij}dx^i dx^j. \tag{45}$$

The metric g_{ab} has also a $3+1$ decomposition with respect to the hypersurface $t = constant$

$$g = N^2dt^2 - h_{ij}(N^i dt + x^i)(N^j dt + dx^j). \tag{46}$$

We define

$$N_j = h_{ij}N^i, \quad \beta^j = \gamma^{jk}\beta_k. \tag{47}$$

Then we have the followings relations

$$N = \hat{\Omega}\tilde{N}, \quad N = \frac{\Omega^2\lambda^2}{(1-\Omega^2\lambda^2\beta_j\beta^j)}, \quad \beta^j = -\frac{N^j}{N^2}, \tag{48}$$

and

$$h_{ij} = \gamma_{ij} - \Omega^2\lambda^2\beta_i\beta_j. \tag{49}$$

Note that

$$N = r^2 f_N, \quad f_N \in E^\omega, \quad f_N(0) = 1. \tag{50}$$

From equation (49), (26), (27), lemma 2.4, lemma 2.5 and theorem 2.1, we can read off the following theorem.

Theorem 2.6. *Assume β given by lemma 2.6. Then, in some neighborhood of i, the metric h_{ij} has the form*

$$h_{ij} = h^1_{ij} + r^3 h^2_{ij}, \tag{51}$$

where h^1_{ij} and h^2_{ij} are analytic.

We remark that $h_{ij} \in W^{4,p}$, $p < 3$ (see e.g. [1] for the definitions of the Sobolev and Hölder spaces $W^{s,p}$ and $C^{m,\alpha}$). This follows from expression (51) and $r^3 \in W^{4,p}$, $p < 3$. It implies, in particular, that the metric is in $C^{2,\alpha}$. With the conformal factor (44), we can define the conformal compactification S of the Cauchy slice $\tilde{S}$ plus the point at infinity i, in the same way as we made for $\tilde{X}$. Theorem 2.6 says that $\tilde{S}, \tilde{h}$ admit a $C^{2,\alpha}$ compactification. Of course, theorem 2.6 only describe the behavior of the fields in a particular foliation. However, happens very unlikely that for other foliations the smoothness improve. The decomposition (40) for β^k, and hence the decomposition (51) for h_{ij}, is preserved under the transformation

$$\beta_k \to \beta_k + \partial_k f, \quad f \in E^\omega. \tag{52}$$

If we impose the condition $x^k \beta_k = 0$, then $\partial_k f$ is fixed. That is, β^k given by lemma 2.5 is the unique vector that satisfies both equation (37) and $x^k \beta_k = 0$.

Let $\tilde{\chi}_{ij}$ be the extrinsic curvature of the hypersurface $\tilde{S}$ with respect to the metric $\tilde{g}_{ab}$. We denote by χ_{ij} the extrinsic curvature of the same hypersurface, but with respect to the conformal metric g_{ab}. These two tensors are related by $\tilde{\chi}_{ij} = \Omega^{-1}\chi_{ij}$. This formula is valid since the conformal factor Ω is independent of the coordinate t. We define the tensor ψ_{ij} by

$$\psi_{ij} = \hat{\Omega}^{-1}\tilde{\chi}_{ij} = \hat{\Omega}^{-2}\chi_{ij}. \tag{53}$$

The tensor ψ_{ij} plays an important role in the conformal method for solving the constraint equations. The following theorem characterizes the behavior of ψ_{ij} near i.

Theorem 2.7. *Assume β given by lemma 2.6. Then, in some neighborhood of i, the tensor ψ_{ij} has the following form*

$$\psi_{ij} = r^{-5} f x_{(i}\beta^2_{j)} + r^{-3}\hat{\psi}_{ij}, \tag{54}$$

where $\hat{\psi}_{ij}, f \in E^\omega$ and the analytic vector β^2_j is given by (41). Moreover, $r^8\psi_{ij}\psi^{ij} \in E^\omega$.

Proof. Since h_{ij} does not depend on t, the extrinsic curvature χ_{ij} satisfies the equation

$$\chi_{ij} = -\frac{1}{2N}\pounds_{N^k} h_{ij}, \tag{55}$$

where $\pounds_{N^k}$ denote the Lie derivative with respect to the vector field N^k. We express this equation in terms of D_i, the covariant derivative with respect to γ_{ij}

$$\chi_{ij} = -\frac{1}{2N}(N^k D_k h_{ij} + 2h_{k(i}D_{j)}N^k) \tag{56}$$

$$= -\frac{1}{2}N\beta^k D_k(\lambda^2\Omega^2\beta_i\beta_j) + \frac{1}{N}h_{k(i}D_{j)}(\beta^k N^2), \tag{57}$$

where, in the second line, we have used equations (48) and (49). Then we use lemma 2.5, equation (50), and equation (53) to prove (54). To compute $r^8\psi_{ij}\psi^{ij}$ we use equation (54) and $x^i\beta_i^2 = 0$. □

Using equation (39) we obtain that ψ_{ij} has a expansion of the form

$$\psi_{ij} = r^{-5}3x_{(i}\bar{\epsilon}_{j)qk}S^q x^k + O(r^{-2}), \tag{58}$$

where S^i is the angular momentum defined by (20).

In general these slices will not be maximal, i.e., $\chi = h^{ij}\chi_{ij} \neq 0$. However, if the space-time is axially symmetric, this foliation can be chosen to be also maximal. In order to prove this, assume that we have an axial Killing vector η^a. The projection η^i of η^a into the hypersurface $\tilde{S}$ is a Killing vector of the metric h_{ij}. We can chose a coordinate system adapted to η^a. In these coordinates we have (see for example [3]) $N_i = \sigma\eta_i$, where $\pounds_\eta\sigma = 0$. Then, using equation (55) and the Killing equation, we obtain

$$\chi_{ij} = -\frac{1}{2N}\eta_{(i}D_{j)}\sigma. \tag{59}$$

From this equation we deduce that $\chi = 0$. The same argument applies, of course, to the physical extrinsic curvature $\tilde{\chi}_{ij}$.

As an application of our results we fill a gap in the proof made in [8] of the following theorem.

Theorem 2.8 (Damour-Schmidt). *Every stationary, asymptotically flat, vacuum space-time admits an analytic conformal extention through null infinity.*

For the definition of null infinity see [15], see also the review [9].

Proof. By lemma 2.5 and equation (25) we have that β_i and λ are analytic with respect to r and the angles x^i/r (but not with respect to x^i!). This proves the assumption made in [8], λ and β_i are essentially the functions F and F_α defined there. □

3 Final Comments

The vector β^k is the essential piece in the translation from the quotient manifold $\tilde{X}$ to the Cauchy slice $\tilde{S}$. This vector is computed as follows. We calculate first the vector J^i by equation (37) in terms of the analytic potentials ϕ_S, ϕ_M and the analytic metric γ_{ij}. The vector J^i is divergence free with respect to the

flat metric. This vector is not analytic in terms of the Cartesian coordinates x^i, since the radial function r appears explicitly in it. The curl of β^k is J^i (cf. equation (37)). In lemma 2.5 we found a solution of this equation which has the desired properties: although is not analytic in x^i it has an analytic expansion in terms of r and the corresponding angles x^i/r. Given the multipole expansion of ϕ_S, ϕ_M and γ_{ij} one can compute the corresponding expansion for β^k with equation (43). Using lemma 2.5 it is straightforward to prove our main result given by theorems 2.6 and 2.7. Those theorems characterize the behavior of the metric and the extrinsic curvature of $\tilde{S}$ near infinity. In particular, we prove that the intrinsic metric h_{ij} is not analytic at infinity, unless we are in the static case.

The Kerr metric is a particular important example of a stationary space time. Explicit computation for this metric has been made in [5], in which we see the behavior described by theorems 2.6 and 2.7.

The condition $r^8\psi_{ij}\psi^{ij} \in E^\omega$ has been used in [7] to construct general initial data which have asymptotic expansion in terms of powers of the radial coordinate. In this work, it has been assumed that the conformal metric is smooth at i. Theorem 2.6 suggest that the same result holds if we only require that the metric has the form (51). This generalization will be studied in a subsequent work [6].

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